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Excessive factorizations of bipartite multigraphs

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ABSTRACT

An *excessive factorization* of a multigraph G is a set $\mathcal{F} = \{F_1, F_2, \dots, F_r\}$ of 1-factors of G whose union is $E(G)$ and, subject to this condition, r is minimum. The integer r is called the *excessive index* of G and denoted by $\chi'_e(G)$. We set $\chi'_e(G) = \infty$ if an excessive factorization does not exist. Analogously, let m be a fixed positive integer. An *excessive* $[m]$*-factorization* is a set $\mathcal{M} = \{M_1, M_2, \dots, M_k\}$ of matchings of G, all of size m, whose union is $E(G)$ and, subject to this condition, k is minimum. The integer k is denoted by $\chi'_{[m]}(G)$ and called the *excessive* $[m]$*-index* of G. Again, we set $\chi'_{[m]}(G) = \infty$ if an excessive $[m]$-factorization does not exist. In this paper we shall prove that, for bipartite multigraphs, both the parameters χ'_e and $\chi'_{[m]}$ are computable in polynomial time, and we shall obtain an efficient algorithm for finding an excessive factorization and excessive $[m]$-factorization, respectively, of any bipartite multigraph.

1. Introduction

In this paper multigraphs are understood to be finite, undirected, without loops and without isolated vertices. Graphs are multigraphs without multiple edges. Let G be a multigraph. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The degree of a vertex v in G is denoted by $\deg_G(v)$. If two vertices x, y are adjacent in G, we shall sometimes denote this by $x \sim y$ or $x \sim_G y$. The symbol xy will denote the set of edges between x and y. If e is an edge joining the vertices x, y, we shall denote this by $e \in xy$. The number of edges joining x and y in G is called the *multiplicity of the edge* xy and is denoted by $\mu_G(xy)$. We shall say that a graph computational problem is solvable in polynomial time (or, simply, is *in* $\mathcal{P}$) if it is solvable in time which is bounded by a polynomial in $|V(G)|$ and $|E(G)|$. For undefined graph-theoretic terminology and notation, we follow Lovász and Plummer [10].

Let G be a multigraph. A 1-*factorization* of G is a set $\mathcal{F} = \{F_1, F_2, \dots, F_d\}$ of edge-disjoint 1-factors (i.e. perfect matchings) of G whose union is $E(G)$. Generalizing this concept, we call the *excessive factorization* of G a set $\mathcal{F}$ of 1-factors of G whose union is $E(G)$ and such that, subject to this condition, $|\mathcal{F}|$ is minimum. Excessive factorizations were introduced by Bonisoli and Cariolaro [1]. The cardinality (i.e. number of 1-factors) of an excessive factorization of G is a graph parameter which we denote by $\chi'_e(G)$ and call the *excessive index* of G. (If no excessive factorization of G exists we set $\chi'_e(G) = \infty$.) Clearly every 1-factorization is an excessive factorization. Moreover, if G is a d-regular multigraph of even order, it is easy to see that G is 1-factorizable if and only if $\chi'_e(G) = d$; hence, as observed in [1], the problem of computing $\chi'_e(G)$ is NP-hard since deciding whether a graph is 1-factorizable is NP-complete [9].

In this paper we shall prove that, for any bipartite multigraph G, the parameter $\chi'_e(G)$ can be computed in polynomial time. For the case when $\chi'_e(G)$ is finite we provide a polynomial time algorithm for constructing an excessive factorization of G.

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The following concept, introduced by Cariolaro and Fu in [2], is a variant of the concept of excessive factorization, where the 1-factors are replaced by matchings of fixed size. Formally, let m be a positive integer. An *excessive* $[m]$*-factorization* of G is a set $\mathcal{M} = \{M_1, M_2, \ldots, M_r\}$ of matchings of G, all of size m, whose union is $E(G)$ and such that, subject to this condition, $|\mathcal{M}|$ is minimum. The cardinality of an excessive $[m]$-factorization (or ∞ if no excessive $[m]$-factorization exists) is called the *excessive* $[m]$*-index* and is denoted by $\chi'_{[m]}(G)$. We call G $[m]$*-coverable* if G admits an excessive $[m]$-factorization, which is equivalent to saying that every edge of G is contained in a matching of size m. It is easy to see that

$$\chi'_{[1]}(G) = |E(G)|,$$

so, trivially, $\chi'_{[1]}$ can be computed in polynomial time.

It was proved in [2] that, for every [2]-coverable graph,

$$\chi'_{[2]}(G) = \max\{\lceil |E(G)|/2\rceil,\ \chi'(G)\}.$$

To prove that we can compute $\chi'_{[2]}(G)$ in polynomial time, we construct an auxiliary graph $H = (E, F)$, where E is the edge set of G and e, f are adjacent vertices in H if and only if e, f are independent edges of G. Now the question "What is the minimum number of matchings of size 2 that covers G?" is reduced to the question "What is the minimum number of edges in an edge cover of H?". It is known that the number of edges in the latter question can be found in polynomial time. Therefore $\chi'_{[2]}(G)$ can be found in polynomial time.

Furthermore, it was shown in [3] that, for every [3]-coverable graph G,

$$\chi'_{[3]}(G) = \max\{\lceil |E(G)|/3\rceil,\ \chi'(G),\ s(G)\}, \tag{1}$$

where $s(G)$ is the maximum cardinality of a set S of edges of G with the property that no pair of distinct edges in S belong to the same matching of size 3 of G. It is not too difficult to see (but we omit the details) that $s(G)$ can be computed in polynomial time. However this does not seem to imply in any obvious way that $\chi'_{[3]}(G)$ can be found in polynomial time, due to the presence of the term $\chi'(G)$ in the right-hand side of (1). Thus the following question of the second author[1] is still open.

> Does there exist a fixed integer (constant) $\mathbf{m}$ such that the complexity of the computation of $\chi'_{[\mathbf{m}]}(G)$ is NP-hard and what is the minimum such integer?

This question will be negatively answered in a forthcoming paper of the authors [6].[2] In this paper we shall prove that, for bipartite multigraphs G, all the excessive $[m]$-indices can be computed in polynomial time. Stated more precisely, we provide an algorithm that, given as input a bipartite multigraph G and any positive integer $m \le \lfloor \frac{|V(G)|}{2} \rfloor$, computes $\chi'_{[m]}(G)$ and, if $\chi'_{[m]}(G)$ is finite, also produces an excessive $[m]$-factorization in time which is bounded by a polynomial in $|V(G)|$ and $|E(G)|$. It will also be shown that, for any bipartite multigraph G, the parameter $\chi'_e(G)$ can be computed in polynomial time.

2. Some preliminary lemmas

Excessive factorizations were first introduced in [1]. In [1] and the subsequent papers on the subject [2,4,3,5] the attention was restricted to (simple) graphs. Lemmas 1–3 were already proved in [2] in the case of graphs, and the proofs therein provided trivially extend to multigraphs. However, both because our arguments provide some simplifications to those used in [2], and in order to keep the exposition self-contained, we include fully detailed proofs of these lemmas. We shall often use a corollary of a well known result of de Werra [7] (proved independently by McDiarmid [11]) which states that a k-edge colourable multigraph with km edges always has a k-edge colouring whose colour classes all have size m (i.e. a decomposition in matchings of size m).

Let the quantity $\Lambda_m(G)$ be defined, for any multigraph G and any positive integer m, as

$$\Lambda_m(G) = \max\{\chi'(G),\ \lceil |E(G)|/m\rceil\}.$$

It is easy to see (see [2]) that

$$\chi'_{[m]}(G) \ge \Lambda_m(G). \tag{2}$$

If G satisfies the equality above, we say that G is *m-compatible*.

We say that a multigraph $\widetilde{G}$ *superlies* on another multigraph G (denoted by $\widetilde{G} \sqsupset G$) if $\widetilde{G}$ is a supergraph of G and u, v are adjacent vertices in $\widetilde{G}$ if and only if they are adjacent vertices in G.

Lemma 1. *Let t be a positive integer. A multigraph G satisfies $\chi'_{[m]}(G) \le t$ if and only if there exists a t-edge colourable multigraph $\widetilde{G} \sqsupset G$ such that $|E(\widetilde{G})| = mt$.*

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[1] This question was first posed at the 21st British Combinatorial Conference in 2007.

[2] Notice that the fact that, for any fixed positive integer m, the parameter $\chi'_{[m]}(G)$ can be computed in polynomial time does not contradict the fact that, in general, the problem of the computation of the excessive index $\chi'_e(G)$ is NP-hard, since the parameter $\chi'_e(G)$ is associated with 1-factors of G, and the size of the 1-factors of G is not fixed but grows with the size of G.

Proof. If $\mathcal{M}$ is an $[m]$-factorization of G with t elements, define a multigraph $\widetilde{G}$ by letting $V(\widetilde{G}) = V(G)$ and by joining the vertices x and y in $\widetilde{G}$ by as many edges as there are matchings $M_i \in \mathcal{M}$ containing an edge of the form xy. Notice that $\widetilde{G} \sqsupset G$. Clearly $|E(\widetilde{G})| = mt$ and, by definition, $\widetilde{G}$ has a decomposition into matchings of size m, whence it is t-edge colourable. The converse is a straightforward consequence of de Werra's theorem. □

Lemma 2. *If the multigraph G satisfies $|E(G)|/m \geq \chi'(G)$, then G is m-compatible, i.e. $\chi'_{[m]}(G) = \lceil |E(G)|/m \rceil$.*

Proof. Let G be a multigraph as in the statement of the lemma. By (2), it will suffice to prove that $\chi'_{[m]}(G) \leq \lceil |E(G)|/m \rceil$. Where φ is any $\chi'(G)$-edge colouring of G, using the fact that $|E(G)|/\chi'(G) \geq m$, we can find a set of m edges all receiving the same colour. Notice that, where G' is the graph obtained from G upon removal of these m edges, then $\chi'(G') \leq \chi'(G)$. Thus, repeating this argument a sufficient number of times, we can cover $\left(\lceil |E(G)|/m \rceil - \chi'(G)\right) m$ edges of G using $\lceil |E(G)|/m \rceil - \chi'(G)$ matchings of size m, leaving at most $\chi'(G)m$ edges uncovered. Let now G'' be any subgraph of G having exactly $\chi'(G)m$ edges and containing all the uncovered edges. In order to terminate the proof it will suffice to show that G'' can be covered by $\chi'(G)$ matchings of size m. This, however, follows immediately from Lemma 1. □

Lemma 3. *Let G be a multigraph. Then there exists an integer $com(G)$ such that G is m-compatible if and only if $1 \leq m \leq com(G)$.*

Proof. Since G is certainly 1-compatible, it clearly suffices to prove that, if G is m-compatible and $m \geq 2$, then G is $(m-1)$-compatible. Assume then that G is m-compatible. We shall prove that

$$\chi'_{[m-1]}(G) = \Lambda_{m-1}(G) = \max\{\chi'(G), \lceil |E(G)|/m-1 \rceil\}. \tag{3}$$

If $|E(G)|/m - 1 \geq \chi'(G)$, this follows from Lemma 2. Hence we can assume that $|E(G)| < (m-1)\chi'(G)$. We show that G can be covered by $\chi'(G)$ matchings of size $m-1$. Since G is m-compatible and $\chi'(G) > |E(G)|/m - 1 > |E(G)|/m$, we have $\chi'_{[m]}(G) = \chi'(G)$. By Lemma 1, there exists a $\chi'(G)$-edge colourable multigraph $\widetilde{G} \sqsupset G$ such that $|E(\widetilde{G})| = \chi'(G)m$. Deleting a sufficient number of edges in $\widetilde{G} - E(G)$, we can obtain a $\chi'(G)$-edge colourable multigraph $\widehat{G} \sqsupset G$ such that $|E(\widehat{G})| = \chi'(G)(m-1)$. By Lemma 1, $\chi'_{[m-1]}(G) \leq \chi'(G)$, which, combined with (2), gives the identity (3), completing the proof. □

For every integer $t \geq \chi'(G)$, we define a function $\zeta_G = \zeta_G(t)$ by letting

$$\zeta_G(t) = \max\{|E(\widetilde{G})| : \widetilde{G} \sqsupset G,\ \chi'(\widetilde{G}) \leq t\}.$$

We have the following.

Lemma 4.

$$\chi'_{[m]}(G) = \min_{t \geq \Lambda_m(G)} \{t : \zeta_G(t) \geq mt\}, \tag{4}$$

where $\min \emptyset$ is defined to be ∞.

Proof. By Lemma 1,

$$\chi'_{[m]}(G) = \min\{t : \exists \widetilde{G} \sqsupset G,\ |E(\widetilde{G})| = mt,\ \chi'(\widetilde{G}) \leq t\}.$$

Clearly there exists a t-edge colourable superlying multigraph of G with mt edges if and only if $|E(G)| \leq mt$, $\chi'(G) \leq t$ and there exists a t-edge colourable superlying multigraph of G with at least mt edges. (Notice that the first two conditions are equivalent to $t \geq \Lambda_m(G)$.) Therefore we have

$$\begin{aligned}\chi'_{[m]}(G) &= \min_{t \geq \Lambda_m(G)} \{t : \exists \widetilde{G} \sqsupset G,\ |E(\widetilde{G})| \geq mt,\ \chi'(\widetilde{G}) \leq t\} \\ &= \min_{t \geq \Lambda_m(G)} \{t : \zeta_G(t) \geq mt\},\end{aligned}$$

which concludes the proof. □

Thus the knowledge of the function $\zeta_G = \zeta_G(t)$ allows us to deduce the exact value of $\chi'_{[m]}(G)$ by means of the formula (4). As a first application of Lemma 4 we now prove that the sequence $\{\chi'_{[m]}(G)\}$ is nondecreasing in m in the interval $[com(G)+1, \infty)$.

Theorem 1. *Let G be a graph. Then the sequence $\{\chi'_{[m]}(G)\}$, for $m \geq com(G)+1$, is nondecreasing in m.*

Proof. Let $m > com(G)$. We prove that $\chi'_{[m]}(G) \leq \chi'_{[m+1]}(G)$. By Lemma 4,

$$\chi'_{[m]}(G) = \min_{t \geq \Lambda_m(G)} \{t : \zeta_G(t) \geq mt\} \tag{5}$$

and

$$\chi'_{[m+1]}(G) = \min_{t \geq \Lambda_{m+1}(G)} \{t : \zeta_G(t) \geq (m+1)t\}. \tag{6}$$

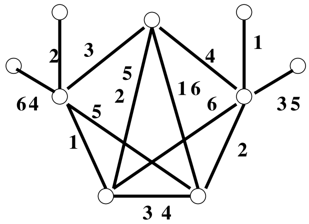

Fig. 1. The graph G in the picture satisfies $\chi'_{[1]}(G) = 13$, $\chi'_{[2]}(G) = 7$, $\chi'_{[3]}(G) = 6 > \max\{\chi'(G), \lceil |E(G)|/3 \rceil\} = 5$; hence $com(G) = 2$. Thus the sequence $\{\chi'_{[m]}(G) : m \geq com(G)\}$ is not monotonic nondecreasing. An excessive [3]-factorization is shown.

Notice that, by Lemma 2 and the assumption that $m > com(G)$ (i.e. that G is not m-compatible), we have

$$\Lambda_m(G) = \Lambda_{m+1}(G) = \chi'(G). \tag{7}$$

Hence, using (5) and (6), in order to conclude the proof it suffices to observe that the set $\{t : \zeta_G(t) \geq mt\}$ contains the set $\{t : \zeta_G(t) \geq (m+1)t\}$. $\square$

We notice that Theorem 1 is best possible since it is not true, in general, that the sequence $\{\chi'_{[m]}(G) : m \geq com(G)\}$ is monotonic nondecreasing, as shown by the example of Fig. 1 (this example is taken from [3]).

3. Bipartite graphs

When G is bipartite, so is every superlying multigraph of G, and, in view of König's Theorem, the definition of the function ζ_G is greatly simplified, since it becomes

$$\zeta_G(t) = \max\{|E(\widetilde{G})| : \widetilde{G} \sqsupset G, \Delta(\widetilde{G}) \leq t\}.$$

The function ζ_G has a natural interpretation; for instance $\zeta_G(\Delta(G)) - |E(G)|$ is exactly the maximum number of edges that can be added to G without creating new adjacencies and without increasing the maximum degree.

We now generalize the definition of the function ζ_G as follows. For any function $f : V(G) \to \mathbb{N}$ satisfying $f(v) \geq \deg_G(v)$ for every $v \in V$, we let

$$\zeta_G(f) = \max\{|E(\widetilde{G})| : \widetilde{G} \sqsupset G, \ \deg_{\widetilde{G}}(v) \leq f(v) \text{ for every } v \in V(G)\}.$$

We shall now reduce the problem of the computation of the function ζ_G defined above to a minimum weight vertex cover (mwvc) problem. Recall that, given a weight function $w : V(G) \to \mathbb{N}$, the mwvc problem asks for a set of vertices W with the property that every edge is incident to at least one vertex in the set and, subject to this condition, the sum of the weights of the vertices of W is minimum. This problem is known to be in $\mathcal{P}$ for bipartite multigraphs. We denote by $mwvc_w(G)$ the value of the minimum weight vertex cover of G with respect to the weight function w.

We are ready to state our main result.

Theorem 2. *Let G be a bipartite multigraph and let $f : V(G) \to \mathbb{N}$ be a function satisfying $f(v) \geq \deg_G(v)$ for every $v \in V$. Then*

$$\zeta_G(f) = |E(G)| + mwvc_w(G),$$

where the weight function w is defined by $w = f - \deg_G$.

Proof. Let (L, R) be a bipartition of G. Define a network N by adding to G a source s, joined to each vertex of L, a sink t, joined to each vertex of R, and orienting all edges from s to L, from L to R and from R to t. Then let, for each arc of the form sx, where $x \in L$, the capacity of sx to be $f(x) - \deg_G(x)$. Similarly, for each arc of the form yt, where $y \in R$, let the capacity of yt be $f(y) - \deg_G(y)$. Let all other arcs (i.e. those joining L to R) have infinite capacity. We claim that the value of a maximum flow in N is precisely the desired quantity $\zeta_G(f)$. For, given a maximum (integer) flow ϕ in N, let, for any pair x, y of adjacent vertices of G,

$$\hat{\phi}(x, y) = \sum_{e \in xy} \phi(e).$$

Define a multigraph $\widetilde{G}$ by letting, for every pair x, y of adjacent vertices of G,

$$\mu_{\widetilde{G}}(xy) = \mu_G(xy) + \hat{\phi}(x, y), \tag{8}$$

and, for every pair of nonadjacent vertices x, y of G, $\mu_{\widetilde{G}}(xy) = 0$. Notice that $\widetilde{G} \sqsupset G$ since $\mu_{\widetilde{G}}(xy) \geq \mu_G(xy)$ if x, y are adjacent in G and $\mu_G(xy) = 0$ if and only if $\mu_{\widetilde{G}}(xy) = 0$. Notice that, for every vertex $u \in L$, we have

$$\begin{aligned}\deg_{\widetilde{G}}(u) &= \sum_{v\sim u}\mu_{\widetilde{G}}(uv) = \sum_{v\sim u}(\mu_G(uv)+\hat{\phi}(u,v))\\ &= \deg_G(u)+\sum_{v\sim u}\hat{\phi}(u,v) = \deg_G(u)+\sum_{v\sim u}\sum_{e\in uv}\phi(e)\\ &= \deg_G(u)+\phi(su)\le \deg_G(u)+(f(u)-\deg_G(u)) = f(u).\end{aligned}\tag{9}$$

Similarly, for every $v \in R$, we have

$$\deg_{\widetilde{G}}(v) \le f(v).\tag{10}$$

Therefore $\widetilde{G}$ satisfies the requirements implicit in the definition of $\zeta_G(f)$, and hence

$$\begin{aligned}\zeta_G(f) \ge |E(\widetilde{G})| &= \sum_{u\in L}\deg_{\widetilde{G}}(u) = \sum_{u\in L}\sum_{v\sim u}\mu_{\widetilde{G}}(uv)\\ &= \sum_{u\in L}\sum_{v\sim u}(\mu_G(uv)+\hat{\phi}(u,v)) = \sum_{u\in L}\sum_{v\sim u}\mu_G(uv)+\sum_{u\in L}\sum_{v\sim u}\hat{\phi}(u,v)\\ &= |E(G)|+\sum_{u\in L}\sum_{v\sim u}\sum_{e\in uv}\phi(e) = |E(G)|+\sum_{u\in L}\phi(su) = |E(G)|+value(\phi).\end{aligned}\tag{11}$$

Conversely, for any multigraph $\widetilde{G} \sqsupset G$ satisfying $\deg_{\widetilde{G}}(x) \le f(x)$ for every $x \in V(\widetilde{G})$, we may construct an integer flow ϕ on N by first selecting, for each multiple edge uv, an edge $e_0 \in uv$, and by letting, for each $e \in uv$,

$$\phi(e) = \begin{cases}\mu_{\widetilde{G}}(uv)-\mu_G(uv) & \text{if } e = e_0\\ 0 & \text{otherwise.}\end{cases}$$

We then extend the definition of ϕ to the arcs of the form su, $u \in L$ and vt, $v \in R$ in such a way as to guarantee conservation of flow, i.e. by letting

$$\phi(su) = \sum_{v\sim u, v\in R}\phi(uv) = \sum_{v\sim u, v\in R}(\mu_{\widetilde{G}}(uv)-\mu_G(uv)) = \deg_{\widetilde{G}}(u)-\deg_G(u)$$

and

$$\phi(vt) = \sum_{u\sim v, u\in L}\phi(uv) = \sum_{u\sim v, u\in L}(\mu_{\widetilde{G}}(uv)-\mu_G(uv)) = \deg_{\widetilde{G}}(v)-\deg_G(v).$$

Notice that this flow satisfies the capacity constraints since

$$\phi(su) = \deg_{\widetilde{G}}(u)-\deg_G(u) \le f(u)-\deg_G(u)$$

and

$$\phi(vt) = \deg_{\widetilde{G}}(v)-\deg_G(v) \le f(v)-\deg_G(v),$$

and the capacity of the arcs of the form uv, where $u \in L$ and $v \in R$, is infinite.

In particular, if $\widetilde{G}$ is chosen in such a way that $|E(\widetilde{G})| = \zeta_G(f)$, we then have

$$value(\phi) = \sum_{u\in L}\phi(su) = \sum_{u\in L}\deg_{\widetilde{G}}(u)-\deg_G(u) = |E(\widetilde{G})|-|E(G)| = \zeta_G(f)-|E(G)|,\tag{12}$$

which, by (11), is a maximum flow. Hence, by (11) and (12), if ϕ is a maximum flow, then the multigraph $\widetilde{G}$ constructed in the first part of the proof satisfies $|E(\widetilde{G})| = \zeta_G(f)$.

By the max-flow min-cut theorem, $\zeta_G(f)$ equals the minimum capacity of an s–t cut of N. Any such cut has the form $(X, \overline{X})$, where $s \in X$ and $t \in \overline{X}$. Let $X_L = X \cap L$, $X_R = X \cap R$, $\overline{X_L} = L \setminus X_L$, $\overline{X_R} = R \setminus X_R$. Then the capacity of $(X, \overline{X})$ is finite if and only if G does not contain any edge of the form uv, where $u \in X_L$ and $v \in \overline{X_R}$, in which case such a capacity is

$$\sum_{x\in\overline{X_L}}(f(x)-\deg_G(x))+\sum_{x\in X_R}(f(x)-\deg_G(x)) = \sum_{x\in\overline{X_L}\cup X_R}(f(x)-\deg_G(x)).$$

We claim that $\overline{X_L} \cup X_R$ is a minimum vertex cover of G. It is a vertex cover since, otherwise, G contains an edge of the form uv, where $u \in X_L$ and $v \in \overline{X_R}$, contrary to the assumption. Moreover, every vertex cover W of G is easily seen to be associated with a cut of capacity $\sum_{w\in W}(f(w)-\deg_G(w))$, i.e. equal to the weight of W under the weight function $f-\deg_G$, and hence has weight greater than or equal to the weight of $\overline{X_L} \cup X_R$ (which is associated with a minimum capacity s–t cut). We conclude that $\zeta_G(f)$ is equal to the minimum capacity of an s–t cut of N, and hence is equal to the minimum weight of a vertex cover of G under the weight function $f-\deg_G$, as desired. □

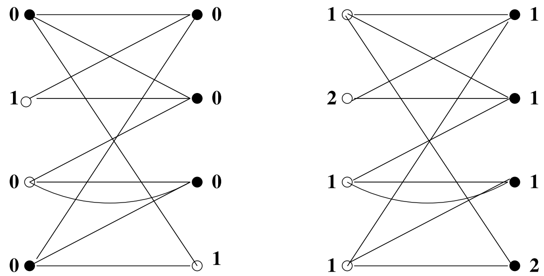

Fig. 2. The bipartite multigraph G of our example (left and right). The integers attached to the vertices represent the weight functions w_0 and w_1. A corresponding minimum weight vertex cover is indicated, for each of them, by vertices in bold.

Corollary 1. *For bipartite multigraphs, and for every f, the function $\zeta_G(f)$ is computable in time $O(|V(G)| \cdot |E(G)| \log |V(G)|)$.*

Proof. It follows from the proof of Theorem 2 and the fact that the max-flow min-cut problem can be solved in $O(|V(G)| \cdot |E(G)| \log |V(G)|)$ by Dinitz's blocking flow algorithm. □

Corollary 2. *$\chi'_{[m]}(G)$ is computable in $O(|V(G)| \cdot |E(G)| \log |V(G)| \log |E(G)|)$ time for bipartite multigraphs.*

Proof. First, we determine whether $\chi'_{[m]}(G)$ is finite. This can be done in polynomial time [8] since it amounts to checking that every edge belongs to a matching of size m. If $\chi'_{[m]}(G)$ is finite then $\chi'_{[m]}(G) \leq |E(G)|$ and, on the basis of Lemma 4, we can assess its precise value by binary search. This requires at most $\log_2 |E(G)|$ queries, each of which costs $O(|V(G)| \cdot |E(G)| \log |V(G)|)$ time. □

Corollary 3. *$\chi'_e(G)$ is computable in $O(|V(G)| \cdot |E(G)| \log |V(G)| \log |E(G)|)$ time for bipartite multigraphs.*

Proof. This follows from Corollary 2 on letting $m = |V(G)|/2$. □

Corollary 4. *An excessive $[m]$-factorization can be found in polynomial time for bipartite multigraphs.*

Proof. First obtain the correct value of $t^* = \chi'_{[m]}(G)$, which can be done in polynomial time by Corollary 2. Then define a network N, as in the proof of Theorem 2, by assigning to the arcs of the form sz or zt a capacity equal to $t^* - \deg_G(z)$ and to the remaining arcs, infinite capacity. Find a maximal flow ϕ of N. Then construct a superlying multigraph $\widetilde{G}$ of G, as in the proof of Theorem 2. Such a multigraph $\widetilde{G}$ is, by Lemma 4, such that

$$|E(\widetilde{G})| = \zeta_G(t^*) \geq mt^* \geq |E(G)|$$

and is t^*-edge colourable because it is bipartite and has maximum degree at most t^* by (9) and (10). Deleting some of the edges of $E(\widetilde{G}) \setminus E(G)$, we obtain a t^*-edge colourable superlying multigraph G' of G with exactly mt^* edges. Find an equalized t^*-edge colouring of G' (i.e. a t^*-edge colouring whose colour classes have all size m), which can be done in polynomial time since G' is bipartite. Let $\varphi : E(G') \to \{1, 2, \ldots, t^*\}$ be such an edge colouring. Then φ can be used to define an excessive $[m]$-factorization $\mathcal{M} = \{M_1, M_2, \ldots, M_{t^*}\}$ of G, by letting the matching M_i contain the edge uv if and only if the colour class $\varphi^{-1}(\{i\})$ contains an edge joining u and v in $\widetilde{G}$. □

Corollary 5. *An excessive factorization can be found in polynomial time for bipartite multigraphs.*

Proof. It suffices to let $m = n/2$ and apply Corollary 4. □

In order to exemplify the concepts just introduced, we now apply our results to the bipartite multigraph G of Fig. 2 and determine the excessive index and an excessive factorization of G. Notice that the size of a perfect matching of G is 4, and hence $\chi'_e(G) = \chi'_{[4]}(G)$.

We have $\Lambda_4(G) = \max\{\chi'(G), \lceil |E(G)|/4 \rceil\} = 3$. We use Lemma 4 and, correspondingly, we evaluate the function $\zeta_G(t)$ for successive values of $t \geq \Lambda_4(G) = 3$, until we obtain the inequality $\zeta_G(t^*) \geq mt^*$, in which case t^* is the required value of $\chi'_{[4]} = \chi'_e(G)$. To evaluate $\zeta_G(3)$ we use Theorem 2 with the function f equal to the constant 3. We have

$$\zeta_G(3) = |E(G)| + mwvc_{w_0}(G),$$

where the weight function $w_0 = 3 - \deg_G$ is the one displayed on the left in Fig. 2. It is easy to see that $mwvc_{w_0}(G) = 0$ (a `mwvc` is displayed in Fig. 2 by bold vertices).

Thus

$$\zeta_G(3) = |E(G)| = 11 < 12 = 4 \cdot 3 = mt,$$

and hence $\chi'_{[4]}(G) > 3$. We now evaluate $\zeta_G(4)$ which, by Theorem 2, is given by

$$\zeta_G(4) = |E(G)| + mwvc_{w_1}(G),$$

where $w_1 = 4 - \deg_G$, as indicated on the right in Fig. 2.

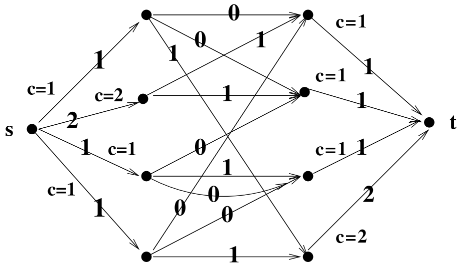

Fig. 3. The network N constructed as in the proof of Theorem 2. A maximum flow and capacities of the arcs are shown. (Where not indicated, the capacities of the arcs are understood to be infinite.)

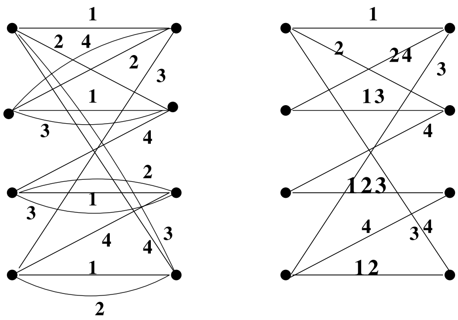

Fig. 4. The multigraph $\widetilde{G}$ resulting from the flow of Fig. 3 (left). A 1-factorization is displayed. The corresponding excessive factorization of G is also displayed (right).

It is possible to see, using the known algorithms for the mwvc or directly, that $mwvc_{w_1}(G) = 5$ (a mwvc is illustrated in Fig. 2 by means of bold vertices).

Hence

$$\zeta_G(4) = |E(G)| + mwvc_{w_1}(G) = 11 + 5 = 16 \geq 4 \cdot 4 = mt^*,$$

and hence, by Lemma 4, $t^* = 4$ is the correct value of $\chi'_{[4]}(G) = \chi'_e(G)$.

To obtain an excessive factorization of G we define a network with source s and sink t, as in the proof of Theorem 2, assigning to each arc of the form sz or zt a capacity equal to $4 - \deg_G(z)$ (i.e. equal to the corresponding weight of the vertex z as indicated at the right of Fig. 2). We then find, e.g. using network flow algorithms or directly, a maximum flow (Fig. 3).

Such a flow is then used to construct a superlying multigraph $\widetilde{G}$ of G by replication of the edges of G according to the value of the flow as indicated in (8) (see Fig. 4). Notice that $\widetilde{G}$ has exactly $16 = mt^*$ edges, and hence coincides with the multigraph G' defined in the proof of Corollary 4. Finally a 1-factorization of $\widetilde{G}$ gives the required excessive factorization of G (see Fig. 4).

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